ISRAEL JOURNAL OF MATHEMATICS, Vol. 28, No. 4, 1977

MODELS WITH FEW ISOMORPHIC EXPANSIONS

BY

A. LITMAN AND S. SHELAH[†]

ABSTRACT

We shall characterize the countable models M, with only countably many expansions by a one place predicate.

The theorem we shall prove here is:

THEOREM 1. *Let M be a countable model, then the following conditions are equivalent*:

(P1) *the number of* (M,P) $(P \subseteq |M|)$ *up to isomorphism is* $\aleph_0$.

(P2) *the number of* (M,P) $(P \subseteq |M|)$ *up to isomorphism is* $< 2^{\aleph_0}$.

(P3) *there are finite models*, N_0, N_1 *such that M is a reduction of a definable expansion of* $N_0 + \sum_{n<\omega} N_1$. (See Definition 0 for the definition of sum of models.)

The consideration of condition (P1) was suggested by Stavi [6], when investigating whether in a Fraenkel–Mostowski model (of set theory) the free Boolean algebra generated by the atoms is a set. He asked whether the number of (M,P) $(P \subseteq |M|)$ up to isomorphism can be $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$ — so the answer is negative.

Clearly if we replace one-place predicate by a two-place predicate, the number of (M,P) is always $2^{\aleph_0}$.

PROBLEM. For a model M of cardinality λ, what can be $|\{(M,P)/\cong : P \subseteq |M|\}|$ and is there a characterization similar to Theorem 1?

Our method is somewhat similar to Shelah [4], and the result was proved by Shelah and announced in [5]. Then Litman shortened the proof by half using the same technique.

[†] The second author is thankful for NSF Grant 43901, by which he was partially supported.
Received April 30, 1975 and in revised form November 29, 1976.

DEFINITION 0. (A) Let $\{M_i \mid i \in I\}$ be an indexed family of models in the same language L such that

1) The models $\{M_i \mid i \in I\}$ have disjoint domains.

2) L contains no functions or constants.

We define $M = \Sigma_{i \in I} M_i$ where M is a model of the same language L, the domain of M is the union of the domains of $\{M_i \mid i \in I\}$ and for any atomic relation R, $M = R\ (a_1, \cdots, a_n)$ iff $a_1, \cdots, a_n$ all belong to the same model, say M_J, and $M_J = R(a_1, \cdots a_n)$.

(B) For two models M_1, M_2 define $M_1 + M_2 = \Sigma_{i \in \{1,2\}} M_i$.

PROOF OF THEOREM 1. It is clear that (P3) $\Rightarrow$ (P1) $\Rightarrow$ (P2), so from now on we shall assume (P2), and eventually prove (P3), by a series of observations.

Let $|M|$ be the universe of M, a, b, c, elements of $|M|$, $\bar{a}, \bar{b}, \bar{c}$ a finite sequence of such elements. Let L be the first-order language associated with M.

$$\operatorname{tp}(a_1, \cdots, a_n) = \{\varphi(x_1, \cdots, x_n) \mid M \vDash \varphi(a_1, \cdots, a_n),\ \varphi \text{ contains no parameters}\}.$$

We shall not distinguich strictly between a sequence $\bar{a}$ and its range.

OBSERVATION 1. Any expansion of M by finitely many individual constants satisfies conditions (P2).

The proof is trivial.

DEFINITION 1. $\langle a_1, a_2, \cdots, a_n \rangle \cong_n \langle b_1, b_2, \cdots, b_n \rangle$ iff $\langle M, a_1, a_2, \cdots, a_n \rangle \cong \langle M, b_1, b_2, \cdots, b_n \rangle$.

OBSERVATION 2. For every n, $\cong_n$ has only finitely many equivalence classes.

PROOF. If $\cong_1$ has infinitely many equivalence classes, we clearly have $2^{\aleph_0}$ non-isomorphic expansions. Assume that the statement is true for $n-1$ and that $\cong_n$ has infinitely many classes, then there is a $n-1$ tuple $\bar{a}$ and an infinite $D \subset |M|$ so that for every pair of distinct members of D $d, d' : \langle \bar{a}, d \rangle \not\cong_n \langle \bar{a}, d' \rangle$. Thus in the model $\langle M, \bar{a} \rangle$ $\cong_1$ has infinitely many classes, contradiction.

OBSERVATION 3. The number of formulas $\varphi(x_0, \cdots, x_n) \in L$ up to equivalence in M is finite. M is homogeneous, and the theory of M is categorical in $\aleph_0$.

PROOF. Simple consequence of Observation 2.

REMARK. As $\operatorname{tp}(\bar{a})$ is equivalent in M to a single formula $\varphi(\bar{x})$, we may assume that $\operatorname{tp}(\bar{a})$ is a single formula.

The following fact will be used implicitly in this paper:

OBSERVATION 4. Let $\Sigma(x_1, \cdots, x_n)$ be a second-order formula in the language of M (i.e., its free second-order relations are atomic relations of M), then there is a first-order formula $\varphi(x_1, \cdots, x_n)$ so that $M \models \forall x_1, \cdots, x_n\ (\varphi \equiv \Sigma)$.

PROOF. Set $Q = \{\mathrm{tp}(a_1, \cdots, a_n) \mid M \models \Sigma(a_1, \cdots, a_n)\}$. Q is finite. Let $\varphi = \bigvee_{\tau \in Q} \tau$. As M is homogeneous, φ is equivalent to Σ.

OBSERVATION 5. There is no formula $\varphi(x, y)$ (φ may contain parameters), so that $\varphi(x, y)$ defines a linear order on any infinite set.

PROOF. Assume $\varphi(x, y)$ is a linear order on some infinite set. We can assume that $\varphi(x, y)$ contains no parameters (otherwise make them individual constants). Let τ be a countable order type. As M is the only model of $\mathrm{Th}(M)$ in $\aleph_0$, there is $A \subset M$ so that $\langle A, \varphi(x, y)\rangle \cong \tau$ (one has to write a diagram which is consistent with $\mathrm{Th}(M)$). As there are $2^{\aleph_0}$ countable order types, M has $2^{\aleph_0}$ expansions. Contradiction.

DEFINITION 2. (A) *A system* is an infinite ordered set $\langle I, < \rangle$, and a finite sequence of functions $F_1, F_2, \cdots, F_n$ s.t. $F_i : I^{k_i} \to |M|$.

(B) Let $\langle I, < \rangle$ be an ordered set and $\bar{\alpha} = \langle \alpha_1, \cdots, \alpha_n \rangle$ a sequence of members of I. Define $\mathrm{atp}(\bar{\alpha}) = \{x_i < x_j \mid \alpha_i < \alpha_j\} \cup \{x_i = x_j \mid \alpha_i = \alpha_j\}$ ($\mathrm{atp}(\bar{\alpha})$ is the set of atomic relations satisfied by $\bar{\alpha}$).

(C) A system $\langle I, < \rangle$, $F_1, F_2, \cdots, F_n$ is *m-homogeneous*, if for any sequences $F_{i_1}, F_{i_2}, \cdots, F_{i_m}$; $\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_m$; $\bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_m$

$$\mathrm{tp}(F_{i_1}(\bar{\alpha}_1), F_{i_2}(\bar{\alpha}_2), \cdots, F_{i_m}(\bar{\alpha}_m)) \neq \mathrm{tp}(F_{i_1}(\bar{\beta}_1), F_{i_2}(\bar{\beta}_2), \cdots, F_{i_m}(\bar{\beta}_m))$$

implies $\mathrm{atp}(\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_m) \neq \mathrm{atp}(\bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_m)$.

(D) Two systems $\langle I, < \rangle, F_1, \cdots, F_n$ and $\langle I', < \rangle, F'_1, \cdots, F'_n$ are *m-similar*, if F_i and F'_i have the same number of places and for any sequences $i_1, i_2, \cdots, i_m \leqq n$; $\bar{\alpha}_1, \bar{\alpha}_2, \cdots \bar{\alpha}_m$; $\bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_m$ so that $\bar{\alpha}_i \subset I$ and $\bar{\beta}_i \subset I'$ if

$$\mathrm{tp}(F_{i_1}(\bar{\alpha}_1), F_{i_2}(\bar{\alpha}_2), \cdots, F_{i_m}(\bar{\alpha}_m)) \neq \mathrm{tp}(F'_{i_1}(\bar{\beta}_1), F'_{i_2}(\bar{\beta}_2), \cdots, F'_{i_m}(\bar{\beta}_m)),$$

then $\mathrm{atp}(\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_m) \neq \mathrm{atp}(\bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_m)$.

(It is clear that if any two systems are m-similar then both are m-homogeneous.)

OBSERVATION 6. Let $\langle I, < \rangle$, $F_1, \cdots, F_n$ be a system, then for any $m < \omega$, there is an infinite $I' \subset I$ so that the system $\langle I', < \rangle$, $F_1 \restriction I', F_2 \restriction I', \cdots, F_n \restriction I'$ is m-homogeneous.

PROOF. By the Ramsey theorem [3] and Observation 2.

OBSERVATION 7. Let $\langle I, <\rangle$, $F_1, \cdots, F_n$ be an m-homogenous system and $\langle I^*, <\rangle$ any countable ordered set, then there are $F_1^*, \cdots, F_n^*$ such that the system $\langle I^*, <\rangle$, $F_1^*, \cdots, F_n^*$ is m-similar to $\langle I, <\rangle$, $F_1, \cdots, F_n$.

PROOF. One has to write a diagram which is consistent with Th(M). As M is the only model of Th(M) in $\aleph_0$, this diagram is realized in M.

DEFINITION 3. (A) "*For almost all* $x \cdots$" will mean "For all x, except for finitely many, $\cdots$"

(B) For $a, b \in |M|$ define $E\ (a, b) =$ For almost all d, $\mathrm{tp}(a, d) = \mathrm{tp}(b, d)$).

DEFINITION 4. Let $A, B \subset |M|$. A and B are *separable* if there is a formula $\varphi(x)$ (φ may contain parameters) s.t. $a \in A \Rightarrow \varphi(a)$ and $b \in B \Rightarrow \sim \varphi(b)$.

OBSERVATION 8. E is an equivalence relation and has finitely many equivalence classes.

PROOF. Clearly, E is an equivalence relation. Assume E has infinitely many classes. Let $\langle Q, <\rangle$ be the rationals. There is a system $\langle Q, <\rangle$, $a_\alpha, b_{\alpha,\beta,\gamma}$ for $\alpha, \beta, \gamma \in Q$ s.t. for any $\alpha \neq \beta$, $\gamma \neq \delta$; $\mathrm{tp}(a_\alpha, b_{\alpha,\beta,\gamma}) \neq \mathrm{tp}(a_\beta, b_{\alpha,\beta,\gamma})$ and $b_{\alpha,\beta,\gamma} \neq b_{\alpha,\beta,\delta}$. From Observations 6 and 7, we can assume that this system is 2-homogenous.

CLAIM. There is a system $\langle \omega, <\rangle$, $c_i, d_{i,j}$, $i, j \in \omega$, such that:

(1) This system is 2-homogenous,

(2) $\mathrm{tp}(c_1, d_{1,0}) \neq \mathrm{tp}(c_0, d_{1,0})$,

(3) For any m, n, $i, j, k \in \omega$, $n \neq m$: $d_{i,n} \neq d_{i,m}$, $\mathrm{tp}(c_k, d_{n,i}) = \mathrm{tp}(c_k d_{n,j})$ and $\mathrm{tp}(d_{m,i}, d_{n,j}) = \mathrm{tp}(d_{m,i}, d_{n,k})$.

PROOF. Let $\alpha_i, i \in \omega$ be an increasing sequence of rationals, and for each $i \in \omega$ let $\beta_{i,k}$, $k \in \omega$ be an increasing sequence of rationals in the interval (α_1, α_{i+1}). Define $c_i^* = a_{\alpha_i}$, $b_{i,j}^* = b_{\alpha_i, \alpha_{i+1}, \beta_{i,j}}$. We have $\mathrm{tp}(c_i^*, b_{i,j}^*) \neq \mathrm{tp}(c_{i+1}^*, b_{i,j}^*)$.

Case 1. $\mathrm{tp}(c_1^*, b_{1,1}^*) \neq \mathrm{tp}(c_0^*, b_{1,1}^*)$. Set $c_i = c_{2i}^*, d_{i,j} = b_{2i,j}^*$.

Case 2. $\mathrm{tp}(c_2^*, b_{1,1}^*) \neq \mathrm{tp}(c_0^*, b_{1,1}^*)$. Set $c_i = c_{2i}^*, d_{i,j} = b_{2i-1,j}^*$. We leave the verification of clauses (1), (2), (3) of the claim to the reader.

Let $\tau = \mathrm{tp}(c_0, d_{0,0})$.

Case 1. The sets $\{c_i \mid i \in \omega\}$, $\{d_{i,j} \mid i, j \in \omega\}$ are separable, by a formula φ ($\varphi(c_i)$ and $\sim \varphi(d_{i,j})$ for all i, j). Let $F: \omega \to \omega$ be any function satisfying $F(i) > \Sigma_{j<i} F(j)$, then there is $A \subset \{c_i \mid i < \omega\} \cup \{d_{i,j} \mid i, j < \omega\}$ such that

$$\{\|\{d : d \in A, \sim \varphi(d), \tau(c,d)\}\| : c \in A \text{ and } \varphi(c)\} = \text{Range}(F).$$

Clearly this is a contradiction to (P2).

Case 2. The sets $\{c_i\}$ and $\{d_{i,j}\}$ are not separable. We can assume that the system $\langle \omega, < \rangle$, $a_i, b_{i,j}$ can be extended to a 2-homogeneous system on $\langle \omega + 1, < \rangle$ (otherwise we take another system by Observation 7). By Observation 5, $\text{tp}(c_0, c_\omega) = \text{tp}(c_\omega, c_0)$, otherwise $\text{tp}(c_0, c_\omega)$ would order the set $\{c_i \mid i < \omega\}$. $tp(c_0, c_\omega) = \text{tp}(c_0, d_{\omega,0})$, or else $\{c_i\}$, $\{d_{i,j}\}$ would be separable by a formula with the parameter c_0. For the same reasons we have

$$\text{tp}(c_0, c_\omega) = \text{tp}(c_\omega, d_{0,0}) = \text{tp}(d_{\omega,0}, d_{0,0}).$$

In conclusion, for any $x, y \in \{c_i\} \cup \{d_{i,j}\}$, if $\tau(x,y)$ then there exists an n s.t. $x, y \in \{c_n, d_{n,i} \mid i < \omega\}$. For any $A \subset \{c_i\} \cup \{d_{i,j}\}$ let $\psi_A(x,y) = (x, y \in A$ and $\tau(x,y)$ or $\tau(y,x))$ and let $\varphi_A =$ the transitive closure of ψ_A. Let E_q be any equivalence relation on ω, then it is possible to construct a set $A \subset \{c_i\} \cup \{d_{i,j}\}$ s.t. $\langle A, \varphi_A \rangle \cong \langle \omega, E_q \rangle$. Contradiction to (P2).

Observation 9. The relation "x is algebraic over y" is an equivalence relation on the non-algebraic elements of M.

Proof. Transitivity and reflexivity are trivial. So assume we have a algebraic over b, b not algebraic over a and a not algebraic. Let $N < \omega$ be such that for all $d \in |M|$, $\langle M, d \rangle$ has less than N algebraic elements (such an N exists by Observation 3). As $\text{tp}(a)$ is not algebraic and E has finitely many classes, there is a sequence $a_1, a_2, \ldots, a_N$ such that $\text{tp}(a_i) = \text{tp}(a)$ and $E\ (a_i, a_j)$ for all $i, j \leqq N$. For almost all d we have $\text{tp}(a_i, d) = \text{tp}(a_j, d)$. As for infinitely many d, $\text{tp}(a_i, d) = \text{tp}(a, b)$, then there is a d such that $\text{tp}(a_i, d) = \text{tp}(a, b)\ \forall i$. Thus, there are at least N algebraic elements over d. A contradiction.

Observation 10. If a is algebraic over $\langle b, c \rangle$ then either a is algebraic over b or a is algebraic over c.

Proof. Assume a is algebraic over $\langle b, c \rangle$ but not algebraic over any single one of them. By Observation 9 (in the models $\langle M, b \rangle$ and $\langle M, c \rangle$) b is algebraic over $\langle a, c \rangle$ and c algebraic over $\langle a, b \rangle$. As $\text{tp}(a)$ is not algebraic, and b not algebraic over a, there is a system $\langle Q, < \rangle$, $a_\alpha, b_{\alpha,\beta}, c_{\alpha,\beta}$ s.t. $\text{tp}(a_\alpha, b_{\alpha,\beta}, c_{\alpha,\beta}) = \text{tp}(a, b, c)$, and for $\gamma \neq \delta$: $a_\gamma \neq a_\delta$ and $b_{\alpha,\gamma} \neq b_{\alpha,\delta}$. By Observations 6, 7 we can assume that this system is 3-homogeneous. For every $\gamma \neq \delta$, $c_{\alpha,\gamma} \neq c_{\alpha,\delta}$, or else we have infinitely many elements algebraic over the pair $\langle a_\alpha, c_{\alpha,\delta} \rangle$. Set $W = \{a_\alpha, b_{\alpha,\beta}, c_{\alpha,\beta} \mid \alpha, \beta \in Q\}$, $\varphi(x,y,z) =$ "x is algebraic over $\langle y, z \rangle$ and $x \neq y \neq z$".

An easy check yields that for distinct $x, y, z \in W$, $\varphi(x, y, z)$ if $\exists \alpha, \beta$, $x, y, z \in \{a_\alpha, b_{\alpha,\beta}, c_{\alpha,\beta}\}$ (in any other case, there will be infinitely many elements algebraic over the same pair). For $A \subset W$ define $\psi_A(xy) =$ "$x, y \in A$ and $\exists z \in A$ $\varphi(x, y, z)$". Let Eq be any equivalence relation on ω, whose equivalence classes have an odd number of elements, then one can build $A \subset W$ s.t. $\langle A, \bar{\psi}_A(x, y)\rangle \cong \langle \omega, \mathrm{Eq}\rangle$ where $\bar{\psi}_A$ is the transitive closure of ψ_A. Contradiction to (P2).

Observation 11. If $E(a, b)$ and d is not algebraic over a and b then $\mathrm{tp}(a, d) = \mathrm{tp}(b, d)$.

Proof. Trivial.

Observation 12. For any a, b, such that neither is algebraic over the other, $\mathrm{tp}(a, b)$ depends only on the E equivalence classes of a and b.

Proof. Let a', b' be another pair s.t. neither is algebraic over the other and $E(a, a')$ and $E(b, b')$. If $a = a'$ then $\mathrm{tp}(a, b) = \mathrm{tp}(a', b')$ by Observation 11. Otherwise, there is a d such that $E(b, d)$ and d not algebraic over a and a' (the E class of b is infinite because b is not algebraic), so we have $\mathrm{tp}(a, b) = \mathrm{tp}(a, d) = \mathrm{tp}(a', d) = \mathrm{tp}(a', b')$.

Definition 5. $E^a(x, y) =$ "For almost all z, $\mathrm{tp}(a, x, z) = \mathrm{tp}(a, y, z)$" (i.e. E^a is the formula E defined in the model $\langle M, a\rangle$).

Observation 13. For nonalgebraic elements a, b and for c nonalgebraic over $\langle a, b\rangle$, $E^c(a, b) \Leftrightarrow E(a, b)$.

Proof. Assume $E(a, b)$ and $\sim E^c(a, b)$, we may assume that a, b are not algebraic one over the other (or else take some d in the same E equivalence class which is not algebraic over a, b, then $\sim E^c(d, b) \cup \sim E^c(d, a)$). By Observation 12, for any d, e which belong to the E equivalence class and are not algebraic one over the other, $\mathrm{tp}(a, b) = \mathrm{tp}(d, e)$. Thus there is a system $\langle Q, <\rangle$, $a_\alpha, c_{\alpha,\beta,\gamma}$ s.t. $\sim E^{c_{\alpha,\beta,\gamma}}(a_\alpha, a_\beta)$, and for $\gamma \neq \delta$: $a_\gamma \neq a_\delta$ and $c_{\alpha,\beta,\gamma} \neq c_{\alpha,\beta,\delta}$. Furthermore, we can assume that this system is 3-homogeneous. Let e be any $c_{\kappa,\lambda,\mu}$. As E^e has finitely many classes, and by the 3-homogeneity, all $a_\sigma, \sigma < \min\{\kappa, \lambda, \mu\}$, are E^e equivalent. Let $0 < \alpha < \beta$, then for infinitely many e, $E^e(a_0, a_\alpha)$ and $\sim E^e(a_0, a_\beta)$ (one of the sets $\{c_{\beta,\beta+1,\gamma} \mid \gamma > \beta + 1\}$ or $\{c_{\frac{1}{2}(\alpha+\beta),\beta,\gamma} \mid \gamma > \beta\}$ will be appropriate). Hence, in the model $\langle M, a_0\rangle$, $\sim E^{a_0}(a_\alpha, a_\beta)$ for $0 < \alpha < \beta$, a contradiction to Observation 8.

Observation 14. For $\bar{a} = \langle a_1, a_2, \cdots, a_n\rangle$ s.t. no a_i is algebraic over the others, $\mathrm{tp}(\bar{a})$ depends only on the E equivalence classes of the a_i.

PROOF. Let $\bar{a}' = \langle a'_1, \cdots, a'_n\rangle$ such that no a'_i is algebraic over the others and $E(a_i, a'_i)$, we have to show that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}')$. Assume the theorem is true for $n-1$. Now if $a'_1 = a_1$, $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}')$ by the induction hypothesis in the model $\langle M, a_1\rangle$ and by Observation 13; otherwise take d which is not algebraic over $\bar{a}, \bar{a}'$ and $E(d, a_1)$. We have $\mathrm{tp}(\bar{a}) = \mathrm{tp}(d, a_2, \cdots, a_n) = \mathrm{tp}(d, a'_2, \cdots, a'_n) = \mathrm{tp}(\bar{a}')$.

DEFINITION 6. (A). Let K be the maximum cardinality of the equivalence classes "x algebraic over y". A sequence $\bar{a}$ is *special* if its length is K and its range is an equivalence class of "x algebraic over y".

(B). $\bar{Z}$ *separates* the two "specials" $\bar{a}, \bar{b}$ if $\mathrm{tp}(\bar{a}, \bar{Z}) \neq \mathrm{tp}(\bar{b}, \bar{Z})$.

(C). Two "specials" $\bar{a}, \bar{b}$ are separable if there is a $\bar{Z}$ (not necessarily special) s.t. $\bar{Z}$ separates $\bar{a}$ and $\bar{b}$, and $\bar{Z} \wedge (\bar{a} \cup \bar{b}) = \emptyset$.

ASSUMPTION. (W.L.O.G.) All the algebraic elements of M are individual constants.

OBSERVATION 15. There is a $\bar{Z}$ which separates any pair of separable "specials".

PROOF. Let $a_i, i < n$ be a sequence of elements s.t. no a_i is algebraic over the others, and each E class contains two elements of this sequence. For $i, j < n$ there is $\bar{Y}_{i,j}$ s.t. for any two "specials" $\bar{a}_i$ which contains a_i and $\bar{a}_j$ which contains a_j, if $\bar{a}_i, \bar{a}_j$ are separable, then $\bar{Y}_{i,j}$ separates them, and $\bar{Y}_{i,j} \wedge (\bar{a}_i \cup \bar{a}_j) = \emptyset$. We may assume that $\bar{Y}_{i,j}$ is algebraicly closed. Let $\bar{s}$ be minimal s.t. $\bar{Y}_{i,j}$ = algebraic closure of $\bar{s}$, then $\mathrm{tp}(a_i, a_j, \bar{s})$ contains the statement "The algebraic closure of $\bar{s}$ separates all separable specials $\bar{b}, \bar{c}$ s.t. $\bar{b}$ contains a_i and $\bar{c}$ contains a_j". Thus by Observation 14 $\bar{Y}_{i,j}$ separates all separable specials $\bar{b}, \bar{c}$ s.t. $\bar{b}$ contain an element which is E-equivalence to a_i, and $\bar{c}$ contain an element which is E-equivalence to a_j. Set $\bar{Z} = \cup \bar{Y}_{i,j}$.

DEFINITION 7. For any "specials" $\bar{a}, \bar{b}$: $E^*(\bar{a}, \bar{b})$ = "$\bar{a}, \bar{b}$ are not separable".

OBSERVATION 16. E^* is an equivalence relation on the special sequence, that has finitely many classes.

PROOF. By Observation 15 E^* has finitely many classes. Thus all we have to show is that E^* is transitive. Assume $E^*(\bar{a}, \bar{b})$, $E^*(\bar{b}, \bar{c})$. It is sufficient to find an $\bar{x}$ s.t. $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{Z})$ ($\bar{Z}$ of Observation 15) and $\bar{x}$ does not separate $\bar{a}$ and $\bar{c}$. Since $\bar{Z}$ contains no algebraic elements, by application of Observation 14 there is $\bar{x}$, $\mathrm{tp}(\bar{x}) = \mathrm{tp}(\bar{Z})$ and $\bar{x} \wedge (\bar{a} \cup \bar{b} \cup \bar{c}) = \emptyset$. As $\bar{x}$ does not separate the pair $\langle \bar{a}, \bar{b}\rangle$ and $\langle \bar{b}, \bar{c}\rangle$, it does not separate the pair $\langle \bar{a}, \bar{c}\rangle$.

OBSERVATION 17. Let $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$ be a sequence of pairwise disjoint "specials", then $\mathrm{tp}(\bar{a}_1, \cdots, \bar{a}_n)$ depends only on the E^* equivalence classes of the $\bar{a}_i$.

PROOF. Let $\bar{b}_1, \cdots, \bar{b}_n$ be another sequence of pairwise disjoint "specials" and $E^*(\bar{a}_i, \bar{b}_i)$. If for all $i \geqq 2$ $\bar{a}_i = \bar{b}_i$ then by definition of E^* $\mathrm{tp}(\bar{a}_1, \cdots, \bar{a}_n) = \mathrm{tp}(\bar{b}_1, \cdots, \bar{b}_n)$, otherwise we can find a sequence of interpolants between $\langle \bar{a}_1, \cdots, \bar{a}_n \rangle$ and $\langle \bar{b}_1, \cdots, \bar{b}_n \rangle$.

THEOREM. *M satisfies* (P3).

PROOF. First let us assume for simplicity that M contains no algebraic elements, and there is one E^*-equivalence class e s.t. any special $\langle a_1, \cdots, a_n \rangle$ has a permutation $\langle b_1, \cdots, b_n \rangle$ which belongs to e. Choose any $\langle b_1, \cdots, b_n \rangle \in e$. Define the model N_1 by $N_1 = \langle \{b_1, \cdots, b_n\}, \equiv, R_i \rangle\, i \leqq n$ where:

1) $b_i \equiv b_J$ for any i, J.

2) $R_i(b_J)$ iff $i = J$.

Clearly M is isomorphic to a reduct of a definable expansion of $\Sigma_{i<\omega} N_1$. If more than one equivalence class of E^* is needed, we get M a definable expansion of $\Sigma_{J=1}^{k}(\Sigma_{i<\omega} N_J) \cong \Sigma_{i<\omega} \bar{N}$ where $\bar{N} = N_1 + N_2 + \cdots + N_k$. If there are algebraic elements, they constitute N_0.

REFERENCES

1. P. Erdos and R. Rado, *Intersection theorems for systems of sets*, J. London Math. Soc. **44** (1969), 467–474.
2. H. J. Keisler, *Infinitary Languages*, North-Holland Publ. Co., Amsterdam, 1971.
3. F. D. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1929), 338–384.
4. S. Shelah, *There are just four second-order quantifiers*, Israel J. Math. **15** (1973), 282–300.
5. S. Shelah, *Various results in mathematical logic*, Notices Amer. Math. Soc. **22** (1975), A–23.
6. J. Stavi, *Free complete Boolean algebras and first order structures*, Notices Amer. Math. Soc. **22** (1975), A-326.

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL

AND

DEPARTMENT OF MATHEMATICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA, USA